

# Calculus Introduction

## Numbers

Positive integers: 1, 2, 3, 4, ...

can add and multiply them  
to get another.

Nonnegative integers: 0, 1, 2, 3, 4, ...

Discovery of 0 was a big deal!

Integers: ..., -3, -2, -1, 0, 1, 2, 3, ...

Now we can also subtract them  
and get another. Now we can be  
in debt! (The American way!)

Rational numbers:  $m/n$ ,  $m = \text{integer}$ ,  
 $n = \text{integer} > 0$ . Now we can divide  
them, except division by 0, and  
get another.

Decimal expansions (we could use  
any base) Examples.

$$\frac{7}{5} = 1.4, \text{ terminates}$$

$$\frac{7}{6} = 1.1666\ldots = : 1.\overline{16}, -\text{means}$$

repeats

$$\begin{array}{r} 1.4 \\ 5 \overline{)7.0} \\ \underline{-5} \\ 20 \\ \underline{-20} \\ 0 \end{array}$$

Fact.  $x$  is rational iff (if and only if) its decimal expansion either terminates or repeats. (For numbers like  $0.\overline{99999\dots} = 0.\overline{9} = 1$  we take the terminating one!).

□ by example.

$$\begin{aligned} x &:= 10.3\overline{36\dots} = 10.\overline{36} \\ 100x &= 1036.\overline{36} \\ x &= 10.\overline{36} \\ \hline 99x &= 1026, \quad x = \frac{1026}{99} \end{aligned}$$

shows a repeating decimal is rational. Same deal in general.

But why is a rational number a terminating or ultimately repeating decimal #?

Remaindering:

$$\frac{m}{n} = q + \frac{r}{n}, \quad 0 \leq r < n.$$

There are only finitely many possibilities for  $r$ . Ultimately the process of long division must repeat, or stop.

$$\begin{array}{r}
 & \overline{10.36} \\
 99 \overline{)1026.0} \\
 99 \\
 \hline
 360 \\
 297 \\
 \hline
 630 \\
 594 \\
 \hline
 360
 \end{array}$$

now it repeats ■

We now extend the rationals to get the real numbers  $\mathbb{R}$ , by including all decimal expansions which do not ultimately repeat.

Here are some important ones:

$$\pi \doteq 3.141592653589793$$

$$e \doteq 2.718281828459046$$

$$g = \frac{1 + \sqrt{5}}{2} \doteq 1.618033988749895,$$

the golden ratio.

These numbers are known to be irrational. There's a better way to test irrationality than above, based on the (extended) Euclidean

algorithm, or the integer-fractional part algorithm. This gives rise to the rcf (regular continued fraction) of a given real number  $x$ . These are of the form

$$x = b_0 + \cfrac{1}{b_1 + \cfrac{1}{b_2 + \cfrac{1}{b_3 + \dots}}}$$

with  $b_0$  an integer, the integer part of  $x$ , and the remaining  $b_k$ 's positive integers. The rcf can terminate, and does iff  $x$  is rational. This is rather clear.

### Example

$$x = \frac{1026}{99} = 10 + \frac{1026 - 10 \cdot 99}{99}$$

$$= 10 + \frac{36}{99} = 10 + \frac{1}{\frac{99}{36}}$$

$$\frac{99}{36} = 2 + \frac{99 - 72}{36} = 2 + \frac{27}{36} = 2 + \frac{3}{\frac{36}{27}}$$

$$\frac{36}{27} = 1 + \frac{36-27}{27} = 1 + \frac{9}{27} = 1 + \frac{1}{3}$$

Thus

$$\begin{aligned} x &= 10 + \frac{1}{99/36} \\ &= 10 + \frac{1}{2 + \frac{1}{36/27}} \\ &= 10 + \frac{1}{2 + \frac{1}{1 + \frac{1}{3}}} \end{aligned}$$

and we have

$$b_0 = 10, b_1 = 2, b_2 = 1, b_3 = 3$$

with termination.

We give some further examples.

$\sqrt{2}$ )

$$\begin{aligned} \sqrt{2} &\approx 1.414 \quad \text{integer part} \\ &= 1 + (\sqrt{2} - 1) \quad \text{fractional part} \\ &= 1 + (\sqrt{2} - 1) \frac{\sqrt{2} + 1}{1 + \sqrt{2}} \\ &= 1 + \frac{1}{1 + \sqrt{2}} \quad \begin{array}{l} \text{reciprocal of} \\ \text{fractional part} \\ \text{has an integer part} \end{array} \end{aligned}$$

But now we can substitute this whole expression in for  $\sqrt{2}$ :

$$\begin{aligned}\sqrt{2} &= 1 + \frac{1}{1 + 1 + \frac{1}{1 + \sqrt{2}}} \\ &= 1 + \frac{1}{2 + \frac{1}{1 + \sqrt{2}}}\end{aligned}$$

Again, and again, ...

$$\begin{aligned}\sqrt{2} &= 1 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \dots}}}}\end{aligned}$$

nonterminating so  $\sqrt{2}$  irrational!

The bks here are (ultimately) periodic with period 1. It's known that such implies that  $x$  is the solution of a quadratic equation with integer coefficients. In this case  $x = \sqrt{2}$  solves  $x^2 = 2$ .

$\sqrt{3})$ 

$$\sqrt{3} \doteq 1.732$$

$$= 1 + (\sqrt{3} - 1) \frac{\sqrt{3} + 1}{1 + \sqrt{3}}$$

$$= 1 + \frac{2}{1 + \sqrt{3}} = 1 + \frac{1}{\frac{1}{2} + \frac{1}{2}\sqrt{3}}$$

$$= 1 + \frac{1}{\frac{1}{2} + \frac{1}{2} \left( 1 + \frac{1}{\frac{1}{2} + \frac{1}{2}\sqrt{3}} \right)}$$

$$= 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \sqrt{3}}}}$$

$$= 1 + \frac{1}{1 + \frac{1}{2 + \frac{1}{1 + \frac{1}{2 + \frac{1}{1 + \dots}}}}}$$

nonterminating and ultimately  
2-periodic  $x = \sqrt{3}$  solves  $x^2 = 3$ .

$\sqrt{4})$   $\sqrt{4} = 2$  terminates, is rational.

$$g) g = \frac{1 + \sqrt{5}}{2} \doteq 1.618$$

$$= 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \dots}}}$$

All  $b_k = 1$ . This is the "slowest converging" rcf but its "approximants"  $1, 2, \frac{3}{2}, \frac{5}{3}, \dots$  do converge to  $g$  reasonably quickly. More about this kind of stuff when we study sequences and series. The decimal expansions are also convergent series. Another "expression" for  $g$  is

$$g = \sqrt{1 + \sqrt{1 + \sqrt{1 + \dots}}}.$$

More precisely this means the sequence

$$g_1 = \sqrt{1} = 1$$

$$g_2 = \sqrt{1 + \sqrt{1}} = \sqrt{2} \doteq 1.414$$

$$g_3 = \sqrt{1 + \sqrt{1 + \sqrt{1}}} = \sqrt{1 + \sqrt{2}} \doteq 1.5538$$

$$g_4 = \sqrt{1 + \sqrt{1 + \sqrt{2}}} \doteq 1.5981$$

⋮

e) 
$$e = 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{4 + \cfrac{1}{1 + \cfrac{1}{6 + \dots}}}}}}$$

very recognizable  
and provable (but not  
by us), pattern, non-  
terminating so  $e$  is irrational  
(best proof).

$\pi)$ 

$\pi = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292 + \cfrac{1}{1 + \cfrac{1}{1 + \ddots}}}}}}$

unrecognizable pattern = no pattern  
 It's known, and easily shown, that the approximants of rcf's alternate, e.g.

$$\pi_0 = 3 < \pi$$

$$\pi_1 = 3 + \frac{1}{7} \doteq 3.1429 > \pi$$

$$\pi_2 = 3 + \frac{1}{7 + \frac{1}{15}} \doteq 3.1415 < \pi$$

$$\pi_3 = 3 + \frac{1}{7 + \frac{1}{15 + \frac{1}{1}}} \doteq 3.1415929 > \pi$$

$$\pi_4 = 3 + \frac{1}{7 + \dots + \frac{1}{292}} \doteq 3.1415926530119 < \pi.$$

The large  $b_4 = 292$  makes this last approximation pretty darn good.

Enough for analytic number theory. But be it known that these basic ideas are related with very serious computational mathematics. They can be related with the fastest

known algorithms for solving large sparse systems of linear equations, typically arising from solving partial differential equations numerically.

Inequalities. The most basic inequality in the WWW (= Whole Wide World) is the agm (arithmetic-geometric mean) inequality. Let  $a > 0$  and  $b > 0$ . Then

$$A(a, b) := \frac{a+b}{2} =: \text{arithmetic mean of } a+b,$$

$$G(a, b) := \sqrt{ab} =: \text{geometric mean of } a+b.$$

Fact.  $A(a, b) > G(a, b)$  unless  $a=b$ , in which case the two means are =.

□ Square both sides to see that it's the same as  $\underbrace{(a+b)^2}_{a^2+2ab+b^2} \geq 4ab$ . But this is the same as  $(a-b)^2 \geq 0$  with = iff  $a=b$ .

There's yet another mean,

$$H(a, b) = \frac{2}{\frac{1}{a} + \frac{1}{b}} = \frac{1}{A(\frac{1}{a}, \frac{1}{b})},$$

the harmonic mean of  
 $a + b$

Since " $A > G$ " then  $\frac{1}{A} < \frac{1}{G}$  so

$$\begin{aligned} H(a, b) &< \frac{1}{G(\frac{1}{a}, \frac{1}{b})} = \frac{1}{\sqrt{\frac{1}{a} \cdot \frac{1}{b}}} = \frac{1}{\sqrt[3]{ab}} \\ &= \sqrt{ab} = G(a, b). \end{aligned}$$

Thus, in general,

$$H(a, b) \leq G(a, b) \leq A(a, b)$$

with equality, in both places,  
iff  $a=b$ .

We can compute  $G(a, b) = \sqrt{ab}$ , i.e.,  
square roots, using the harmonic-  
arithmetic mean algorithm

Start with  $0 < a_0 \leq b_0$

for  $n = 0, 1, 2, \dots$

$$\begin{cases} a_{n+1} = H(a_n, b_n) \\ b_{n+1} = A(a_n, b_n) \end{cases}$$

Try it with  $a_0 = 1$ ,  $b_0 = 2$ . You'll like it. Here are the results:

$n$	$a_n$	$b_n$
0	1.	2.
1	1.3	1.5
2	1.4117647...	1.416
3	1.414211438...	1.414215686...
4	1.414213562371500	1.414213562374690
5	1.414213562373095	1.414213562373095

$$\hat{=} \sqrt{2} = G(a_0, b_0)$$

These results are similar with those of Gauss who, at age 14, used 20 digit computations of the geometric-arithmetic mean algorithm. He could compute the geometric mean because that's what the harmonic-arithmetic mean algorithm converges, very fast, to.

The theory of the "ham" algorithm is easy, that of "gam" gives rise to fast algorithms for computing elliptic functions, among many many others. It must be easy too, but I don't know it (yet).

## Quadratic functions.

$a, b, c$  real,  $a \neq 0$

$$y = f(x) := ax^2 + bx + c, \quad -\infty < x < +\infty.$$

Complete the square to learn all about all such quadratics.

$$\begin{aligned} y = f(x) &= a \left( x^2 + \frac{b}{a}x + \frac{c}{a} \right) \\ &= a \left[ \left( x + \frac{b}{2a} \right)^2 + \frac{c}{a} - \frac{b^2}{4a^2} \right] \\ &= a \left( x + \frac{b}{2a} \right)^2 + c - \frac{b^2}{4a} \\ &= a \left( x + \frac{b}{2a} \right)^2 + \frac{4ac - b^2}{4a} \\ &= \underbrace{a \left( x + \frac{b}{2a} \right)^2}_{\geq 0} - \frac{b^2 - 4ac}{4a} \end{aligned}$$

①  $a > 0 \Rightarrow$

$$f(x) \geq c - \frac{b^2}{4a} \text{ with}$$

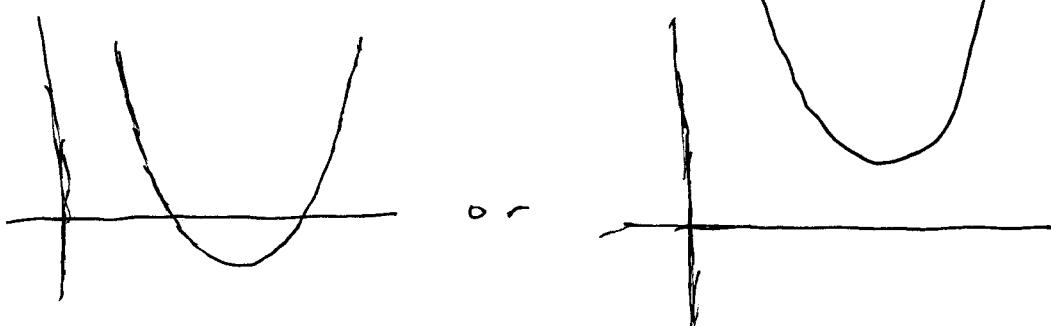
$$\text{equality iff } x = -\frac{b}{2a}$$

$f$  has a global minimum at

$$x_{\min} := -\frac{b}{2a} \text{ with minimum value}$$

$$f(x_{\min}) = c - \frac{b^2}{4a} \leq c.$$

Picture.



$a > 0$  means  $f$  is "bowl-shaped up" (= convex). Example:  $y = x^2$ !

②  $a < 0 \Rightarrow$

$$f(x) \leq c - \frac{b^2}{4a} \text{ with}$$

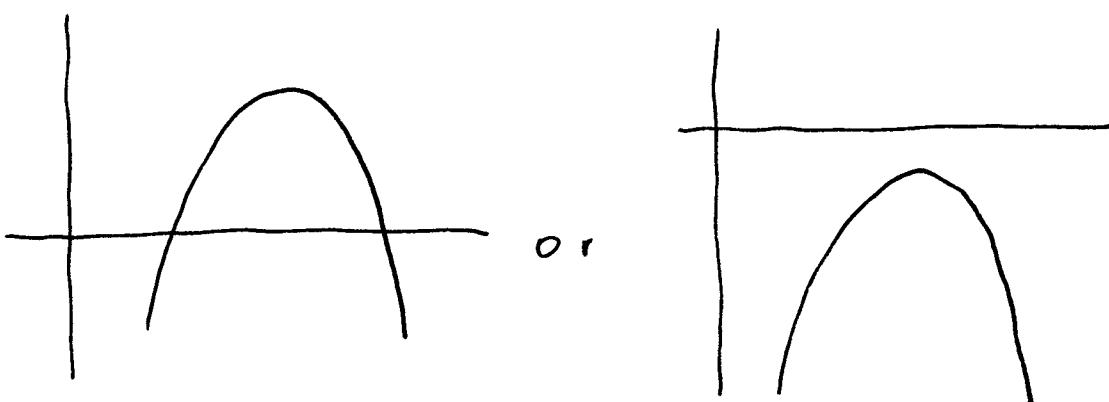
equality iff  $x = -\frac{b}{2a}$

$f$  has a global maximum at

$x_{\max} := -\frac{b}{2a}$  with maximum

value  $f(x_{\max}) = c - \frac{b^2}{4a} \geq c$ .

Picture



## Quadratics (continued).

③ Zeros of f.  $f(x) = 0 \Leftrightarrow$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

$\Leftrightarrow$

$$x + \frac{b}{2a} = \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$\Leftrightarrow$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

the "high school formula". If  $b^2 \geq 4ac$  there are two real zeros of f, which coalesce at  $x = -b/2a$  when  $b^2 = 4ac$ . If  $b^2 < 4ac$  f has no real zeros. See the pictures. But it then has two nonreal complex zeros which are conjugates of each other. See the appendix on complex numbers. These two zeros are (z for complex!)

$$z = \frac{-b \pm i\sqrt{b^2 - 4ac}}{2a},$$

with  $i^2 = -1$ ,  $i$ : imaginary unit.

The high school formula is numerically unstable in the case of two real zeros. (Most "teachers" don't know that!)

Fact. The only arithmetic operation, of  $\times, /, +$  and  $-$ , which can cause a significant increase in relative (rounding) errors is subtraction (of nearly equal #'s).

□ The computer holds about 16 decimal digit numbers. If two, perfectly rounded but not exact, numbers are subtracted, and their first 8 digits are the same, then the result is accurate to only 8 digits. Half the precision has been lost. ■

We don't want to be more rigorous here, but Calculus can be used to firm up these matters. I compute, therefore I am. I always try to remove all possible

cancellation from an algorithm! <sup>(I/17)</sup>

In

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \quad b^2 \geq 4ac,$$

there are two sources of possible cancellation. The first one can be removed. The other, if it occurs (i.e.  $4ac > 0$ ), is inherent to the problem and cannot be removed. We first note Vieta's formulas. From

$$\begin{aligned}f(x) &= ax^2 + bx + c \\&= a(x - x_0)(x - x_1) \\&= a[x^2 - (x_0 + x_1)x + x_0x_1]\end{aligned}$$

we have, on equating coefficients of like powers of  $x$ ,

$$x_0 + x_1 = -\frac{b}{a}, \quad x_0x_1 = \frac{c}{a}$$

Thus if  $x_0$  can be computed with minimal cancellation then  $x_1$  can be computed as

$$x_1 = \frac{c}{ax_0},$$

with only multiplicative operations, hence no subtractions.

And we may take

$$x_0 := \frac{-b - \sqrt{b^2 - 4ac}}{4a}, \quad b \geq 0$$

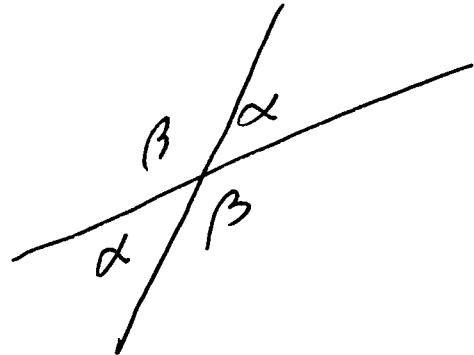
$$:= \frac{-b + \sqrt{b^2 - 4ac}}{4a}, \quad b < 0.$$

So the only cancellation in this algorithm is the inherent one  $b^2 - 4ac$ , if there is any at all.

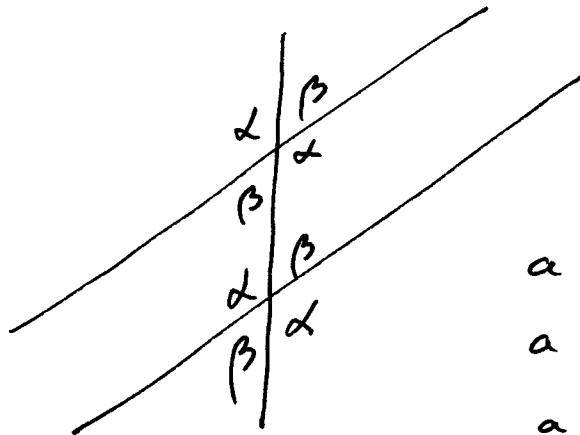
(I can even compute  $b^2 - 4ac$  to high relative precision, always, but that's a more interesting story.)

### High School Geometry.

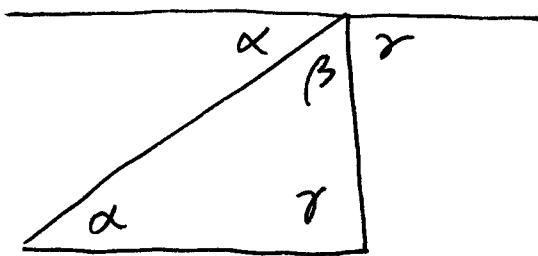
Some basic facts - with pictures.



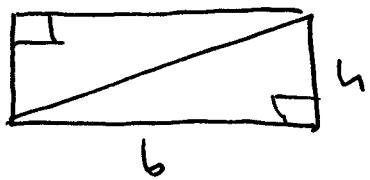
"Vertical angles  
are equal".



If two parallel lines are cut by a transversal then alternate interior angles are =.



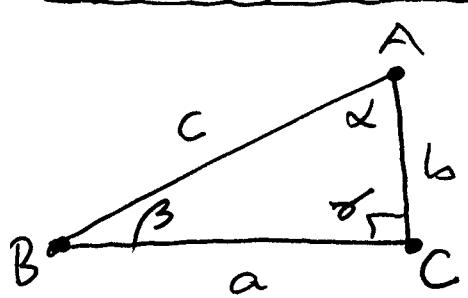
The sum of the angles of a triangle is a straight angle ( $180^\circ$ )



The area of a right triangle is  $\frac{1}{2}$  its base  $\times$  its height

Problem. The same is true for any triangle.

Pythagorean theorem

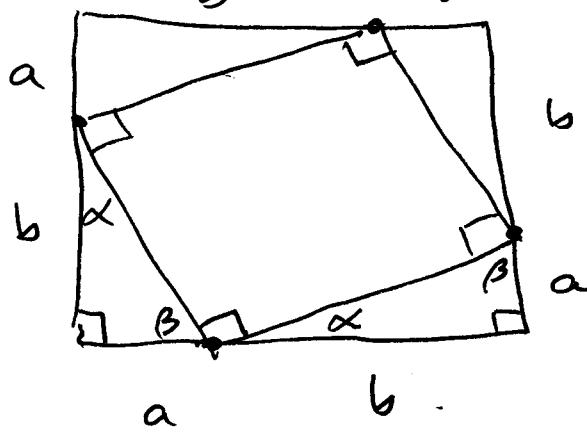


In any right triangle, with  $c$  the length of the hypotenuse,

$$a^2 + b^2 = c^2$$

Corollary. distance(A, B) =  $\sqrt{a^2 + b^2}$

□ Squares  $\cong$



Equate areas

$$(a+b)^2 = c^2 + \\ + 4 \cdot \frac{1}{2} ab$$



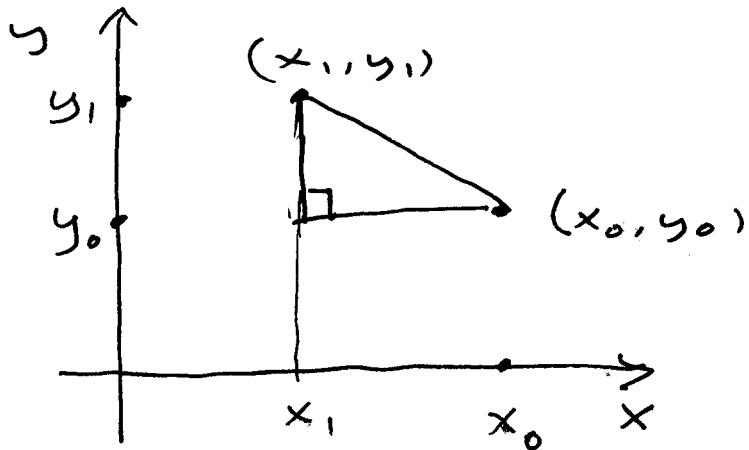
$$a^2 + 2ab + b^2 = c^2 + 2ab$$



$$a^2 + b^2 = c^2 \blacksquare$$

The proof uses geometry & algebra.

Same corollary, with coordinates.



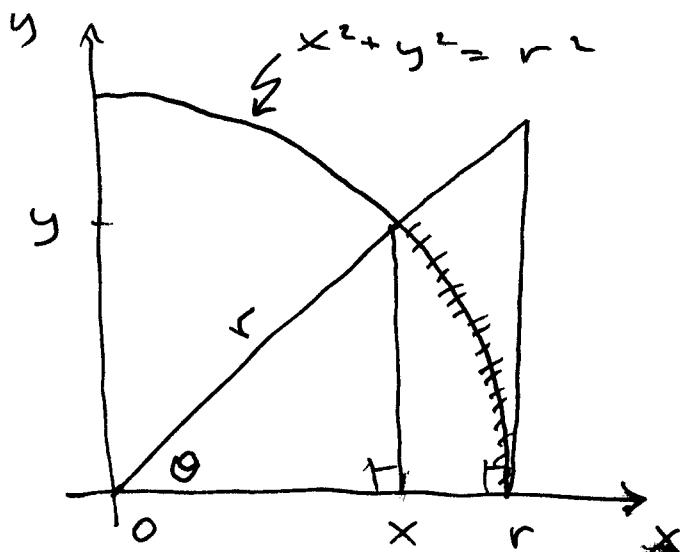
distance between  $(x_0, y_0)$  &  $(x_1, y_1)$

$$= \sqrt{(x_0 - x_1)^2 + (y_0 - y_1)^2}$$

This extends to 3, 4, ..., n dimensions. See the text for n=3.  
(Don't fuss with n=4, just believe!)

## Trigonometry.

### The big picture



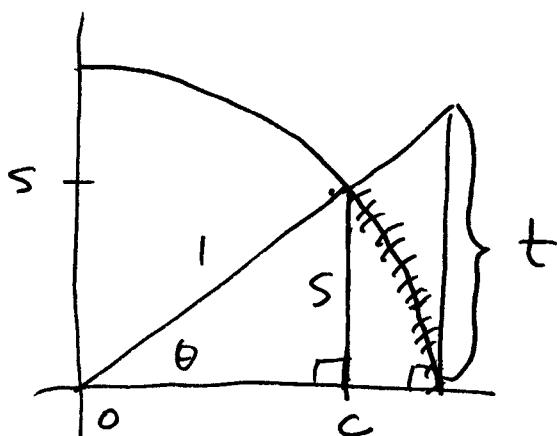
$$x = r \cos \theta$$

$$y = r \sin \theta$$

$$\frac{y}{x} = \tan \theta$$

$(r, \theta)$  are polar coordinates.

Special case  $r=1$  (unit circle)



Pythagoras

$$c^2 + s^2 = 1$$

Similar  $\Delta$ s

$$\frac{t}{s} = \frac{1}{c}, \text{ i.e.}$$

$$t = \frac{s}{c}$$

### Abbreviations

$$c = \cos \theta, s = \sin \theta, t = \tan \theta$$

Radian measure  $\theta = 2\pi r$

= arc length

= length of " "

Archimedes gave an algorithm for computing  $\pi$  to arbitrarily high accuracy. He inscribed and circumscribed regular polygons with  $3 \cdot 2^n$  sides in the unit circle and, using only the Pythagorean theorem, found a recursive way to compute their perimeters, and areas. This is the beginning of the Integral Calculus.

Besides radians, the other usual method for measuring angle is to use degrees. There are  $360^\circ$  in a full circle, so

$$360^\circ = 2\pi \text{ radians}$$

$$180^\circ = \pi \text{ radians}$$

$$1^\circ = \frac{\pi}{180} \text{ radians}$$

$$1 \text{ radian} = \frac{180}{\pi} \text{ degrees.}$$

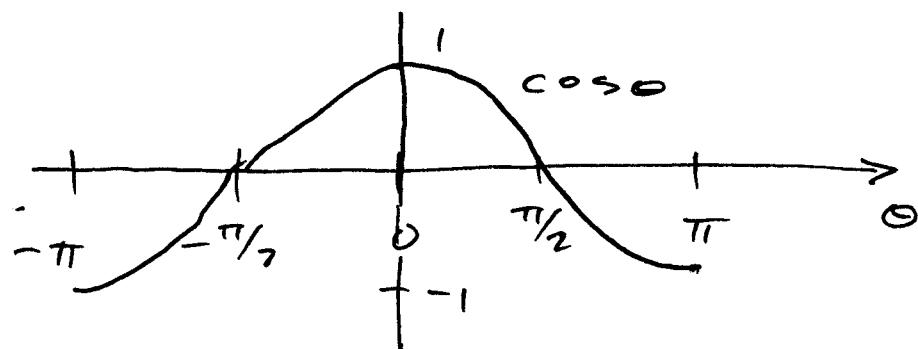
But why  $360^\circ$ . A clock has 3600 seconds - irrelevant. Answer: the "nicest" polygon is a hexagon,

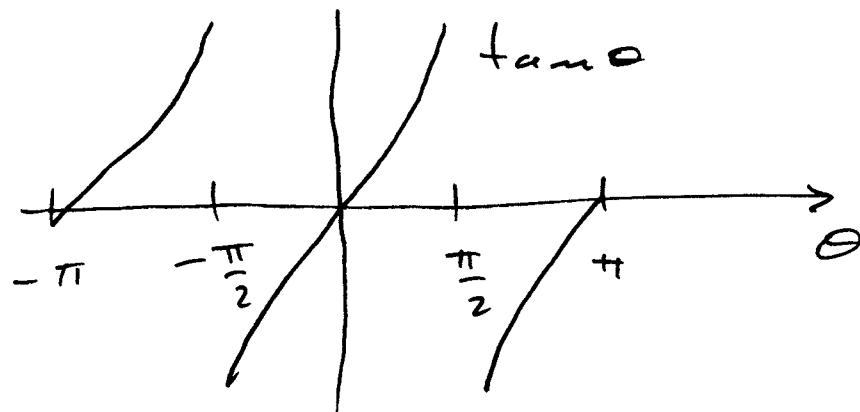
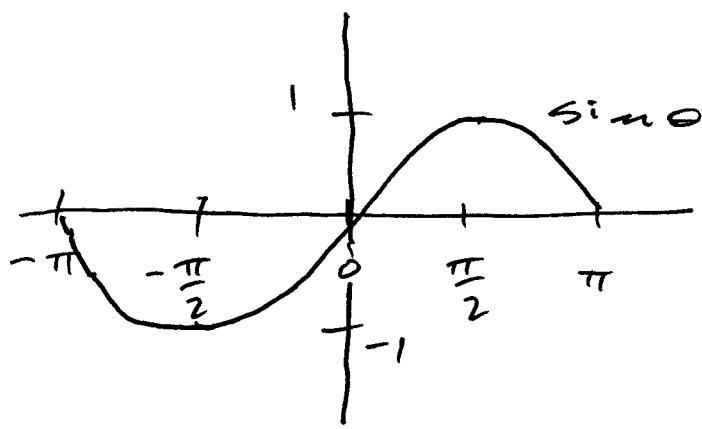
built from 6 equilateral triangles with side 1. Thus 3 is a nice lower bound for  $\pi$ . So the circle was split into 6 arcs. And the Babylonians used base 60.  $6 \cdot 60 = 360^\circ$ .

The Indiana legislature once tried to pass a law that  $\pi = 3 !!$

$\cos\theta$ ,  $\sin\theta$  and  $\tan\theta$  are periodic with period  $2\pi$  radians =  $360^\circ$ ,  $\cos\theta$  is even,  $\sin\theta$  and  $\tan\theta$  are odd. (See text for definitions)

Rough graphs.

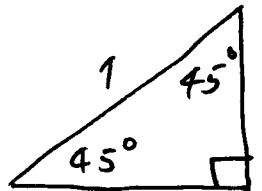




Actually,  $\tan \theta$  has period  $\pi$ !  
Some special values

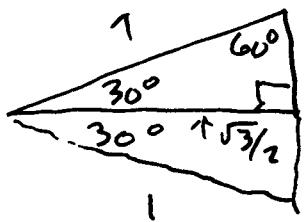
$\theta^{\circ}$	$\cos \theta$	$\sin \theta$	$\tan \theta$
0	1	0	0
15	-	-	-
30	$\sqrt{3}/2$	$1/2$	$\sqrt{3}/3$
45	$\sqrt{2}/2$	$\sqrt{2}/2$	1
60	$1/2$	$\sqrt{3}/2$	$\sqrt{3}$
75	-	-	-
90	0	1	$\pm\infty$

because



$$s, c = s = \frac{\sqrt{2}}{2}$$

by  $c = s$  & Pythagoras.



$$c = \frac{\sqrt{3}}{2}, s = \frac{1}{2}$$

by equilateral triangles + Pythagoras.

But what are the numbers for  $15^\circ + 75^\circ = 90^\circ - 15^\circ$ ?

The area theorem (used later to compute the derivatives of the trig functions).

$$\cos\theta < \frac{\sin\theta}{\theta} < \frac{1}{\cos\theta},$$

$$0 < |\theta| < \frac{\pi}{2}$$

- . There are three triangles in the big trig picture, one being a circular triangle. Since the area of the whole disk is  $\pi$ , the area of the circular triangle is  $\frac{\theta}{2\pi} \cdot \pi = \frac{\theta}{2}$ . (We're taking  $0 < \theta < \frac{\pi}{2}$ )

here, temporarily.) The small triangle has area  $\frac{1}{2}cs$ , the large one has area  $\frac{1}{2}(1 + \frac{s}{c}) = \frac{1}{2}\frac{s}{c}$ . (It's base is of unit length.)

Thus, cancelling  $\frac{1}{2}$ ,

$$\cos\theta \sin\theta < \theta < \frac{\sin\theta}{\cos\theta},$$

$$0 < \theta < \frac{\pi}{2}.$$

Divide by  $\theta > 0$ :

$$\underbrace{\cos\theta}_{\text{even}} \underbrace{\frac{\sin\theta}{\theta}}_{\text{even}} < 1 < \underbrace{\frac{1}{\cos\theta}}_{\text{even}} \underbrace{\frac{\sin\theta}{\theta}}_{\text{even}},$$

$$0 < |\theta| < \frac{\pi}{2}.$$

Now divide the first inequality by  $\cos\theta$  and multiply the second by  $\cos\theta$  to get the result. ■

Corollary.

$$\frac{\sin\theta}{\theta} \rightarrow 1 \text{ as } \theta \rightarrow 0$$

□ Because  $\frac{\sin\theta}{\theta}$  is "squeezed" between  $\cos\theta + \frac{1}{\cos\theta}$  and those

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(continuous functions) both  $\rightarrow 1$   
as  $\theta \rightarrow 0$

### The basic trig identities

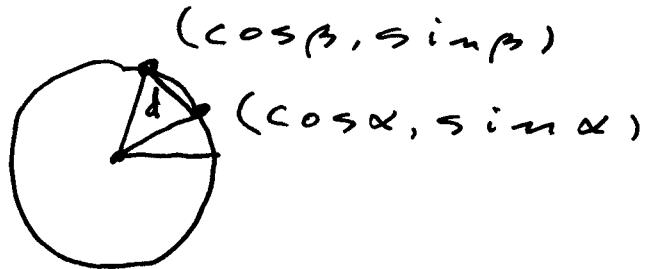
The most basis is the Pythagorean identity

$$\cos^2 \theta + \sin^2 \theta = 1$$

↑

means identically  
equal with  
(for all  $\theta$ ).

### Cosine subtraction formula

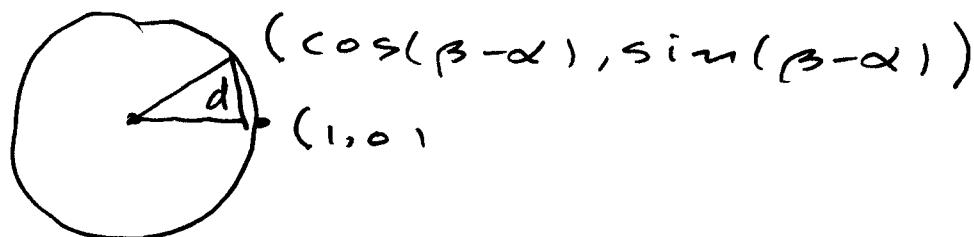


Let  $(\cos \beta, \sin \beta)$  and  $(\cos \alpha, \sin \alpha)$  be two points on the unit circle, with angles  $\beta$  and  $\alpha$ , respectively. Let  $d$  be the distance between them. Then

$$d^2 = (\cos \beta - \cos \alpha)^2 + (\sin \beta - \sin \alpha)^2$$

$$\begin{aligned}
 d^2 &= \cos^2 \beta - 2 \cos \alpha \cos \beta + \cos^2 \alpha \\
 &\quad + \sin^2 \beta - 2 \sin \alpha \sin \beta + \sin^2 \alpha \\
 &= 2 - 2 (\cos \alpha \cos \beta + \sin \alpha \sin \beta)
 \end{aligned}$$

Now rotate the two points through  $-\alpha$  to get the new picture



It's a rigid rotation so  $d$  remains the same. But it is now

$$\begin{aligned}
 d^2 &= (\cos(\beta-\alpha) - 1)^2 + \sin^2(\beta-\alpha) \\
 &= \cos^2(\beta-\alpha) - 2\cos(\beta-\alpha) + 1 \\
 &\quad + \sin^2(\beta-\alpha) \\
 &= 2 - 2 \cos(\beta-\alpha) \\
 &= 2 - 2 \cos(\alpha-\beta)
 \end{aligned}$$

by Pythagoras and the evenness of  $\cos \theta$ .

Hence

$$\boxed{\cos(\alpha - \beta) = \cos\alpha \cos\beta + \sin\alpha \sin\beta}.$$

Now swap  $\beta$  and  $-\beta$ , and use that  $\cos\beta$  is even and  $\sin\beta$  is odd:

$$\boxed{\cos(\alpha + \beta) = \cos\alpha \cos\beta - \sin\alpha \sin\beta}$$

Two very special cases

$$\begin{aligned}\cos(\alpha + \pi/2) &= \cos\alpha \cancel{\cos\pi/2}^0 - \sin\alpha \cancel{\sin\pi/2}^0 \\ &= -\sin\alpha\end{aligned}$$

$$\begin{aligned}\cos(\alpha + \pi) &= \cos\alpha \cancel{\cos\pi}^{-1} - \sin\alpha \cancel{\sin\pi}^0 \\ &= -\cos\alpha.\end{aligned}$$

You can see these in the plots.

Now

$$\begin{aligned}\sin(\alpha + \beta) &= -\cos(\alpha + \beta + \pi/2) \\ &= -\cos\left((\alpha + \frac{\pi}{2}) + \beta\right) \\ &= -\left[\cos(\alpha + \frac{\pi}{2}) \cos\beta - \sin(\alpha + \frac{\pi}{2}) \sin\beta\right] \\ &= \underbrace{\sin\alpha \cos\beta + \sin(\alpha + \frac{\pi}{2}) \sin\beta}_{-\cos(\alpha + \pi)} \\ &= -\cos(\alpha + \pi) = \cos\alpha\end{aligned}$$

$$\boxed{\sin(\alpha+\beta) = \cos\alpha \sin\beta + \sin\alpha \cos\beta}.$$

The last two boxed formulas are the trig addition formulas for the cosine and sine.

They are easy to remember. For this, define

$$\boxed{\text{cis } \theta := \cos\theta + i \sin\theta, \quad i^2 = -1},$$

cis being for "cosine plus i sine".

From the formula for multiplication of complex numbers,

$$(a+ib)(c+id) = (ac-bd) + i(ad+bc),$$

we find

$$\text{cis } \alpha \text{ cis } \beta = (\cos\alpha + i \sin\alpha).$$

$$\begin{aligned} & \cdot (\cos\beta + i \sin\beta) = \\ & = (\cos\alpha \cos\beta - \sin\alpha \sin\beta) + \\ & \quad + i(\cos\alpha \sin\beta + \sin\alpha \cos\beta) \\ & = \cos(\alpha+\beta) + i \sin(\alpha+\beta), \end{aligned}$$

that is

$$\boxed{\text{cis } (\alpha+\beta) = \text{cis } \alpha \text{ cis } \beta}.$$

We shall later use power series to show that, in a very "real" sense (yuk, yuk!),

(I3)

$$\text{cis } \theta = e^{i\theta}$$

We shall then have Euler's formulas

$$e^{i\theta} = \cos \theta + i \sin \theta$$

$$e^{i\alpha} e^{i\beta} = e^{i(\alpha+\beta)}$$

The latter is the law of exponents for imaginary arguments.  $\alpha$  and  $\beta$  are real so  $i\alpha$  and  $i\beta$  are imaginary. Setting  $\theta = \pi$  in Euler's first formula gives

$$e^{i\pi} = -1$$

This relates four important numbers of mathematics,  $-1$ , the first negative #,  $\pi$  the number of trig, and Calculus via Archimedes, ~~and~~  $e$ , the base of the "natural" logarithm, why?

and i, the main new number in the powerful subject of complex variables.

Addition formula for tangent.

$$\boxed{\tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}}$$

□

$$\begin{aligned}\tan(\alpha + \beta) &= \frac{\sin(\alpha + \beta)}{\cos(\alpha + \beta)} = \\ &= \frac{\cos \alpha \sin \beta + \sin \alpha \cos \beta}{\cos \alpha \cos \beta - \sin \alpha \sin \beta}.\end{aligned}$$

Divide numerator and denominator by  $\cos \alpha \cos \beta$  and use  $\tan \theta = \sin \theta / \cos \theta$ . ■

Problem. Show that  $\tan \theta$  is  $\pi$ -periodic.

Double angle formulas. Set  $\alpha = \beta = \theta$  in the above formulas, and use Pythagoras, to get

$$\begin{aligned}\cos 2\theta &= \cos^2 \theta - \sin^2 \theta \\ &= 2\cos^2 \theta - 1 \\ &= 1 - 2\sin^2 \theta\end{aligned}$$

$$\sin 2\theta = 2 \cos \theta \sin \theta$$

$$\tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta}$$

### Half angle formulas.

Use the last two formulas in the first boxed set just above, and replace  $\theta$  by  $\theta/2$ , to get

$$\cos \frac{\theta}{2} = \pm \sqrt{\frac{1 + \cos \theta}{2}},$$

$$\sin \frac{\theta}{2} = \pm \sqrt{\frac{1 - \cos \theta}{2}}.$$

Because  $C^2 + S^2 = 1$  we know  $|C| \leq 1$  and  $|S| \leq 1$ . The latter formulas can involve cancellation, and that's not nice. But I won't try to fix it now. (If one involves cancellation the other doesn't, just an observation.) What about  $\tan$ ? Set

$$t := \tan \theta, \quad \tau = \tan \frac{\theta}{2}$$

and use the above formula to get

$$t = \frac{2\tau}{1-\tau^2},$$

that is

$$\tau\tau^2 + 2\tau - t = 0,$$

or

$$\tau^2 + \frac{2}{t}\tau - 1 = 0$$

By the quadratic formula

$$\begin{aligned} \tau &= \frac{-\frac{2}{t} \pm \sqrt{\frac{4}{t^2} + 4}}{2} \\ &= -\frac{1}{t} \pm \sqrt{1 + \frac{1}{t^2}} \\ &= -\frac{1}{t} \pm \frac{1}{t} \sqrt{1 + t^2} \\ &= \frac{-1 \pm \sqrt{1+t^2}}{t} \quad \frac{-1 \mp \sqrt{1+t^2}}{-1 \mp \sqrt{1+t^2}} \\ &= \frac{1 - (1+t^2)}{t(-1 \mp \sqrt{1+t^2})} \\ &= \frac{t}{1 \pm \sqrt{1+t^2}} \end{aligned}$$

If the + sign is a correct one then

$$\boxed{\tan \frac{\theta}{2} = \frac{\tan \theta}{1 + \sqrt{1 + \tan^2 \theta}}},$$

an incredibly beautiful, and seemingly useful, formula,  
since no cancellation.

Problem. For which values of  $\theta$  is it true? (Ans:  $|\theta| < \frac{\pi}{2}$ ? )

Finding the missing values.

$$\cos 15^\circ = ?, \sin 15^\circ = ?, \tan 15^\circ = ?$$

□  $\cos^2 \frac{\theta}{2} = \frac{1+\cos\theta}{2}, \sin^2 \frac{\theta}{2} = \frac{1-\cos\theta}{2}$

$$\cos 30^\circ = \frac{\sqrt{3}}{2}, \sin 30^\circ = \frac{1}{2}, \tan 30^\circ = \frac{\sqrt{3}}{3}$$

$$\cos^2 15^\circ = \frac{1 + \frac{\sqrt{3}}{2}}{2} = \frac{2 + \sqrt{3}}{4}$$

$$\boxed{\cos 15^\circ = \frac{\sqrt{2+\sqrt{3}}}{2}} \quad (2\sqrt{ } \leq !)$$

$$\sin^2 15^\circ = \frac{1 - \frac{\sqrt{3}}{2}}{2} = \frac{2 - \sqrt{3}}{4} \cdot \frac{2 + \sqrt{3}}{2 + \sqrt{3}}$$

$$= \frac{1}{4(2 + \sqrt{3})}$$

$$\boxed{\sin 15^\circ = \frac{1}{2\sqrt{2+\sqrt{3}}}}$$

These check  
with matlab!

$$15^\circ = \frac{\pi}{12} \text{ radians!}$$

$$\boxed{\tan 15^\circ = \frac{1}{2 + \sqrt{3}}}$$

$$c = \cos \theta, s = \sin \theta, t = \tan \theta$$

In  $|\theta| < \pi/2$ ,

$$c > 0, \text{ signs} = \text{sign } t$$

Always

$$c^2 + s^2 = 1, t = \frac{s}{c}.$$

So

$$s = t c, c^2(1+t^2) = 1$$

$$c = \frac{1}{\sqrt{1+t^2}}, s = \frac{t}{\sqrt{1+t^2}},$$

$$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

No cancellation here! Maybe this is generally useful, a principle?